GOOGLE CODE-IN 2012	Google Code-In 2012–BRL-CAD: Some Basic Affine Transformations		
Document No.	N/A	Author	Matt S.
Revision	0.0	Date	Dec. 2012

Rev.	Author	Date	Summary of Changes
0.0	Matt S.	01/12/2012	Document created.

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1 Some Basic Affine Transformations

A lot more well-written information is available on the Internet, so I won't go into any real detail here. Instead, the interested reader may refer to this primer, among many others.

This article will simply provide formulae to accomplish specific tasks.

1.1 Affine Translation of Points

Assume we have a collection of discreet points $\{x_i\} \subset \mathbb{R}^3$ that we want to rigidly translate in such a way that a specific point x_0 is translated to the origin, thus preserving the relative placement of all points.

To do this, create a vector

$$b = -\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

and then create the augmented matrix

$$A_T = \begin{bmatrix} I_3 & -b \\ 0 & 1 \end{bmatrix}$$

so that, for each x_i, we compute

$$\begin{bmatrix} y_i \\ 1 \end{bmatrix} = \begin{bmatrix} I_3 & -b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ 1 \end{bmatrix}$$

where the y_i represent the translated x_i.

1.2 Affine Rotation About an Axis

The cheat that we have performed here is that by first translating all points of interest to the origin, we may now rotate about a Cartesian axis to make all our points coincident to a Cartesian plane; let's say the $x^1 - x^2$ plane.

First, we take three points $\{y_0, y_1, y_2\} \subseteq \{y_i\}$ and determine the normal via

$$n = y_1 \times y_2$$

since $y_0 = 0$ now, thus allowing us treat the coordinates y_1 and y_2 as vector elements in the computation of n. This allows us to determine the angle φ between n and y^3 via

$$\cos(\varphi) = \frac{n \cdot y^3}{\|n\|}$$

where we treat y^3 as a unit vector $(0, 0, 1)$. Then, compute a unit normal vector to the plane defined by span $\left(n, y^3\right)$ via

$$\begin{aligned} u =& n \times y^3 \\ \Rightarrow u_\mu =& \frac{1}{\|u\|} u \end{aligned}$$

so that we may align n and y^3 via the rotation

$$R = I_3 \cos\varphi + \sin\varphi \, [u_\mu]_\times + (1 - \cos\varphi)\, u_\mu \otimes u_\mu$$

where

$$[u_\mu]_\times = \begin{bmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{bmatrix}$$

$$u_\mu \otimes u_\mu = \begin{bmatrix} \left(u^1\right)^2 & u^1 u^2 & u^1 u^3 \\ u^1 u^2 & \left(u^2\right)^2 & u^2 u^3 \\ u^1 u^3 & u^2 u^3 & \left(u^3\right)^2 \end{bmatrix}$$

Finally, this allows us to then define an augmented (rotation) matrix

$$A_R = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

allowing us to rotate our collection of points $\{y_i\}$ into the $y^1 - y^2$ plane via

$$\begin{bmatrix} z_i \\ 1 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_i \\ 1 \end{bmatrix}$$

and we may now very easily compute the area of the polygon defined by the points z_i via Green's Theorem.