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13. ABSTRACT (Maximum 200 words)

The goal was to study the use of (smooth) piecewise polynomial spaces for the approximation of functions in one and, preferably, in several variables and find a better understanding of how well one can approximate from specific spaces, for specific schemes for approximation, including good bases for such spaces, and to make inroads on the problem of extending techniques for curve fitting by smoothly patched curves to surface interpolation.

Progress was made on these questions, and on two related but unanticipated projects: (i) a monograph on box splines (to make available in easily accessible form the many results on box splines obtained by us and others since their introduction by us ten years ago); and (ii) what now looks like the "right" approach to polynomial interpolation in several variables.

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Major results include the following:

(a) The surprising fact that smoothness requirements restrict the approximation power of bivariate piecewise polynomials much less than was assumed in the finite element literature.

(b) A thorough understanding of the mechanism of quasi-interpolation for the construction of optimal approximation schemes .

(c) A surprisingly simple and effective algorithm for rational spline curves .

(d) The use of ideal theory and harmonic analysis for the understanding of the polynomials contained in a box spline space an investigation which ultimately led to the most important result of this grant, the new approach to multivariable polynomial interpolation .

(e) Algorithms and programs for the construction of C^1 surfaces consisting of polynomial patches and fitting to a given arbitrary triangular mesh of data or curves.

ARO 24074.15-MA

Spline Functions and Surfaces

Final Report

Carl de Boor

November 26, 1990

for the
U. S. Army Research Office
DAAL03-87-K-0030

Center for the Mathematical Sciences
University of Wisconsin-Madison

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A. STATEMENT OF THE PROBLEMS STUDIED We started out with the goal of studying the use of (smooth) piecewise polynomial spaces for the approximation of functions in one and, preferably, in several variables. We were looking for a better understanding of how well one can approximate from specific spaces, for specific schemes for approximation, including good bases for such spaces, and hoped to make inroads on the problem of extending techniques for curve fitting by smoothly patched curves to surface interpolation.

While we made progress on these questions, we also pursued two related but unanticipated projects: (i) a monograph on box splines (to make available in easily accessible form the many results on box splines obtained by us and others since their introduction by us ten years ago); and (ii) what now looks like the 'right' approach to polynomial interpolation in several variables.

B. SUMMARY OF THE MOST IMPORTANT RESULTS This brief outline relies on the fact that more details can be found in the semi-annual reports submitted during the grant period, and, if need be, in the manuscripts filed with ARO as required. All boldfaced numbers refer to items in the list of publications given in C.

Major results include the following:

(a) The surprising fact that smoothness requirements restrict the approximation power of bivariate piecewise polynomials much less than was assumed in the finite element literature (see **5**, **12**).

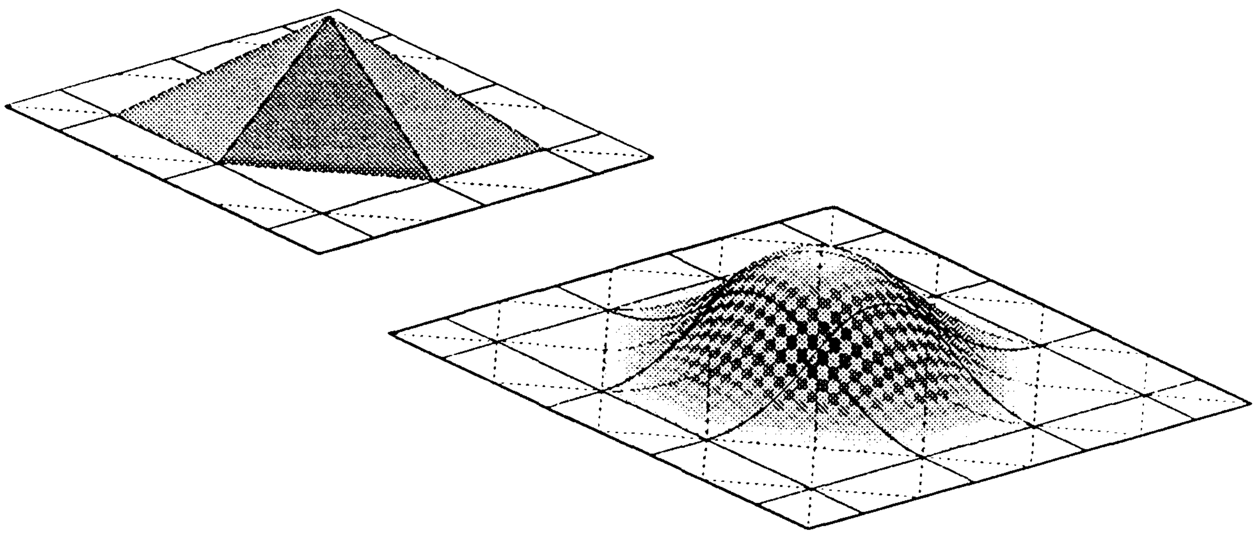
(b) A thorough understanding of the mechanism of quasi-interpolation for the construction of optimal approximation schemes (see **11**, **17**, **18**).

(c) A surprisingly simple and effective algorithm for rational spline curves (see **2**).

(d) The use of ideal theory and harmonic analysis for the understanding of the polynomials contained in a box spline space (see **9**, **11**, **13**, **14**), an investigation which ultimately led to the most important result of this grant, the new approach to multivariable polynomial interpolation (see below).

(e) Algorithms and programs for the construction of C^1 surfaces consisting of polynomial patches and fitting to a given arbitrary triangular mesh of data or curves.

(f) Some of the research findings (e.g., **1**, **3**, **15**) were the result of questions which arose during our work on a **book on box splines**. This monograph, jointly authored by C. de Boor, K. Höllig and S. Riemenschneider (and admittedly much longer in the making than we anticipated when we started three years ago), is now close to completion, missing only some final work on the last three chapters and notes on the relationship of the material presented to the existing literature. The chapter headings (1. Box splines defined; 2. The linear algebra of box spline spaces; 3. Quasiinterpolants and approximation order; 4. Cardinal interpolation and difference equations; 5. Cardinal splines and cardinal series; 6. Discrete box splines and linear diophantine equations; 7. Subdivision algorithms) provide a feeling for the material covered, but do not hint at the many illustrations provided nor at the great effort made in other ways to present the basic box spline theory in as simple and as illuminating a way as possible (but see Figures 1 and 2 below, as well as **16**, where the material of chapter 7, restricted to the practically important two-dimensional case, is quoted). This book has occupied much of de Boor's research time (and of Höllig's while he was still supported by this grant). Part of the difficulty in writing this book was due to the fact that, in the exuberance of initial discovery, many of the authors of the basic papers in this theory were not all that careful in their presentation. Also, with the hindsight afforded by such a systematic study of the literature, we found ourselves confronted with various questions left open, yet needing to be answered in a comprehensive discussion. The book is to be published by Springer-Verlag. We hope that its publication will further the use of box splines in CAGD and in the developing multivariable wavelet theory.



(1)Figure. The hat function and the Zwart element

(g) The most important research finding obtained by work supported by this grant is the realization that there is a very simple way of assigning, to each finite pointset Θ in $\mathbb{R}^s$, a polynomial space Π_Θ suitable for interpolation to arbitrary data on that pointset. The initial discovery is reported in **8**, numerics of the scheme are addressed in **19**, **20**, and a far-reaching generalization to interpolation, not just to point values but to arbitrary linear information, is detailed in **21**.

Polynomial interpolation is at the basis of most of the local approximation that goes into the construction of quadrature and differentiation rules, and, on a more sophisticated level, into numerical schemes for the solution of functional equations, be they integral or differential equations. But, while this basic construct is fully understood in the univariate case (and has been for centuries, as is attested by the fact that such names as Newton, Cauchy, Lagrange, and Hermite are associated with it), the multivariate story is completely different. The basic difficulty has been to decide just what polynomial space to use to interpolate from when given data at a certain set Θ of points in $\mathbb{R}^s$. Even if the cardinality of Θ happens to coincide with the dimension of the space Π_k of all polynomials of degree $\leq k$ in s variables, there is no reason to believe that such interpolation is possible. Thus past efforts have mostly been directed at discovering specific pointsets Θ at which interpolation from Π_k is possible. (A first effort to provide reasonable polynomial interpolation at arbitrary pointsets (Kergin interpolation) provided the clue needed to find the recurrence relations for simplex splines, thus setting off an investigation into multivariate splines that continues unabated to this day.) In contrast, we were able to work out a way of associating with every finite pointset Θ in $\mathbb{R}^s$ a polynomial space Π_Θ from which one can uniquely interpolate at the points of Θ. Now, that by itself is not so impressive. What is remarkable are the various attributes our choice Π_Θ has:

(i) The choice is monotone, i.e., $\Theta \subset \Theta'$ implies that $\Pi_\Theta \subset \Pi_{\Theta'}$. This makes it possible to construct a Newton form for the interpolant, i.e., the introduction of an additional data point only requires a modification of the interpolant already constructed.

(ii) The choice is continuous (to the extent that that is possible), i.e., small changes in Θ usually don't change Π_Θ much (if at all).

(iii) As points coalesce (in a disciplined way), we obtain Hermite (or osculatory) interpolation in the limit.

(iv) Each Π_Θ is translation-invariant, hence differentiation-invariant.

(v) Each Π_Θ is scale-invariant, hence spanned by homogeneous polynomials.

(vi) For any invertible matrix A and any point $c \in \mathbb{R}^s$, $\Pi_{A\Theta+c} = \Pi_\Theta \circ A^T$. This implies that Π_Θ inherits any symmetries (such as invariance under rotations and/or reflections) that Θ might have. In conjunction with (v), it says that Π_Θ is unchanged if Θ is shifted and/or scaled.

(vii) Among all polynomial interpolants to given data on Θ, our interpolant has the smallest possible degree.

(viii) If Θ is a cartesian product (e.g., a grid of points), then Π_Θ is a tensor product of the polynomial spaces assigned to the (lower-dimensional) factor sets.

(ix) A polynomial vanishes at every point of Θ if and only if the constant coefficient homogeneous differential operator associated with its leading term vanishes on Π_Θ. For example, if all the points of Θ lie on some circle in the plane, then Π_Θ consists of harmonic polynomials.

While we came upon it in an investigation of spaces which are limits of exponential spaces (as the frequencies coalesce, see **8**), we found subsequently (see **19**, **20**) that our particular choice Π_Θ can also be arrived at more directly by an interesting variation on Gauss elimination applied to the Vandermonde matrix (ϑ^α) for the points $\vartheta \in \Theta$. (I hasten to add that it seems unlikely that this scheme would have been found by experimenting with that Vandermonde matrix without the benefit of knowing what we wanted to find.)

C. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS REPORTING PERIOD, INCLUDING JOURNAL REFERENCES:

1. C. de Boor, K. Höllig and S. Riemenschneider, Fundamental solutions for multivariate difference equations, *Amer.J.Math.* **111** (1989), 403–415.

2. K. Höllig, Algorithms for rational spline curves, *Trans. Fifth Army Conf. Applied Mathematics Computing*, ARO Report 88-1, pp.287–300

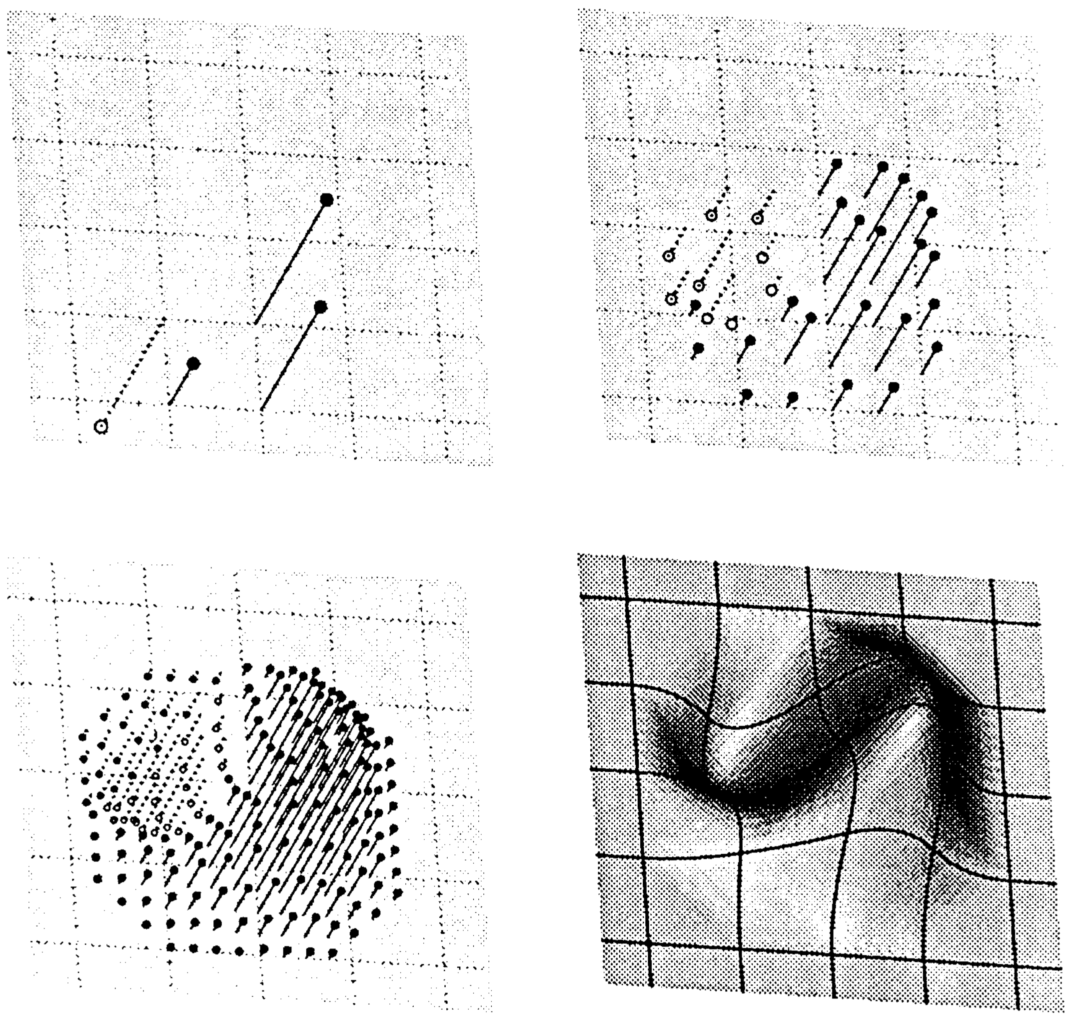
3. C. de Boor & K. Höllig, Minimal support for bivariate splines, *J. Approximation Theory Appl.* **3** (1987), 11–23.

4. C. de Boor, The exact condition number of the B-spline basis may be hard to determine, *J.Approximation Theory* **60** (1990), 344–359.
5. C. de Boor and K. Höllig, Approximation power of smooth bivariate pp functions, *Math. Z.* **197** (1988), 343-363.
6. C. de Boor, What is a multivariate spline?, in *Proc. First Intern. Conf. Industr. Applied Math., Paris 1987*, J. McKenna and R. Temam eds., 1988, 90-101.
7. C. de Boor, B(asic)-spline basics, 27 pages, in *Pioneering Works in CAD*, L. Piegl ed., Butterworth, 1990. To appear
8. C. de Boor and A. Ron, On multivariate polynomial interpolation, *Constructive Approximation* **6** (1990), 287–302.
9. C. de Boor and A. Ron, 'The limit at the origin of a smooth function space', in *Approximation Theory V*, C.K.Chui, L.L. Schumaker & J. Ward eds., Academic Press, New York, 1989, 93–96.
10. C. de Boor, Local corner cutting and the smoothness of the limiting curve, *Computer Aided Geometric Design* **7** (1990), 389–397.
11. C. de Boor and A. Ron, On polynomial ideals of finite codimension with applications to box spline theory, *J.Mathem. Anal. Applic.* **xx** (199x), xxx–xxx.
12. C. de Boor, A local basis for certain smooth bivariate pp spaces, in *Multivariate Approximation Theory IV*, W. Schempp & K. Zeller eds., Birkhäuser, Basel, 1989, 25–30.
13. C. de Boor, N. Dyn and A. Ron, On two polynomial spaces associated with a box spline, *Pacific J. Math.* **xx** (199x), xxx–xxx.
14. C. de Boor and A. Ron, Polynomial ideals and multivariate splines, in *Multivariate Approximation Theory IV*, W. Schempp & K. Zeller eds., Birkhäuser, Basel, 1989, 31–40.
15. C. de Boor and K. Höllig, Box-spline tilings, *Amer. Math. Monthly* **xx**, (199x), xxx-xxx.
16. K. Höllig, Box-Spline Surfaces, in *Mathematical Methods in Computer Aided Geometric Design*, Tom Lyche & L.L.Schumaker eds., Academic Press, 1989, 385–402.
17. C. de Boor, Quasiinterpolants and approximation power of multivariate splines, in *Computation of curves and surfaces*, W. Dahmen, M. Gasca and C.A. Micchelli eds., Kluwer Academic Publishers, Dordrecht, Netherlands, 1990, 313–345.
18. C. de Boor and A. Ron, The exponentials in the linear span of integer translates of a compactly supported function, *J.London Math.Soc.* **xx** (199x), xxx-xxx.
19. C. de Boor, Polynomial interpolation in several variables, Proceedings of the Conference honoring Samuel D. Conte, R. DeMillo and J.R. Rice eds., Plenum Press, 199x, xxx–xxx.
20. C. de Boor and A. Ron, Computational aspects of polynomial interpolation in several variables, *Math.Comp.* submitted.
21. C. de Boor and A. Ron, The least solution for the polynomial interpolation problem, *Math.Z.* submitted.

D. LISTING OF ALL PARTICIPATING SCIENTIFIC PERSONNEL SHOWING ANY ADVANCED DEGREES EARNED BY THEM WHILE EMPLOYED ON THE PROJECT

C. de Boor (entire time), K. Höllig (from 1 July 87 to 1 January 89), A. Ron (from 27 August 89 to 26 September 90).

(The grant provided no student support, but a student, Jörg Peters, completed a thesis (and several papers and reports) on the material mentioned in B.(e).)



(2)Figure. The mesh functions a, $a^{1/2}$, $a^{1/4}$ as well as the corresponding box spline surface obtained as limit of subdivision.